Kazhdan-Laumon Categories and Representations

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1. Background

- We work with a split reductive group G over a finite field $\mathbb{F}_q$ with Weyl group W. Kazhdan-Laumon categories were defined in 1988 by Kazhdan and Laumon ([1]) by gluing tuples of perverse sheaves on the basic affine space G/U indexed by the Weyl group W. This gluing is done in analogy to constructing the abelian category of sheaves on a variety by gluing the categories of sheaves on an open cover. In this analogy, transition functions between open sets are replaced by symplectic Fourier-Deligne transforms $\Phi_w : G/U \to G/U$ indexed by Weyl group elements.
- Kazhdan and Laumon planned to use the Grothendieck groups of these categories to provide a new geometric construction of discrete series representations of $G(\mathbb{F}_q)$, but their construction relied on their conjecture that these categories each had finite cohomological dimension, which was found to be false by Bezrukavnikov and Polishchuk in [2] (even for the case $G = \mathrm{SL}_3$).
- This poster is a summary of three recent projects which seek to (1) better understand Kazhdan-Laumon categories in the context of modern representation-theoretic objects, (2) understand symplectic Fourier transforms and their connection to knot/braid theory, and (3) carry out Kazhdan and Laumon's original construction by modifying their conjecture in an appropriate way.

2. Kazhdan-Laumon Categories

- **Definition.** The Kazhdan-Laumon category $\mathcal{A}$ associated to G is the abelian category of tuples $(A_w)_{w\in W}$ of perverse sheaves on G/U indexed by the Weyl group equipped with "compatible" morphisms $\Phi_s A_w \to A_{sw}$ for any $s \in S$, $w \in W$.
- **Definition.** The Kazhdan-Laumon category $\mathcal{O}$, which we call $\mathcal{A}_B$, is the full subcategory of tuples of B-equivariant perverse sheaves on G/U.
- For any $w \in W$, let $P(w)$ denote the standard parabolic subgroup of W generated by the simple reflections $s \in S$ for which $l(ws) > l(w)$.

  Theorem 1. (M-F., [3]) Any simple object in $\mathcal{A}_B$ is a tuple such that for some fixed $w \in W$ and left coset $P(w)z$, its yth entry is IC_w if $y \in P(w)z$ and 0 otherwise.

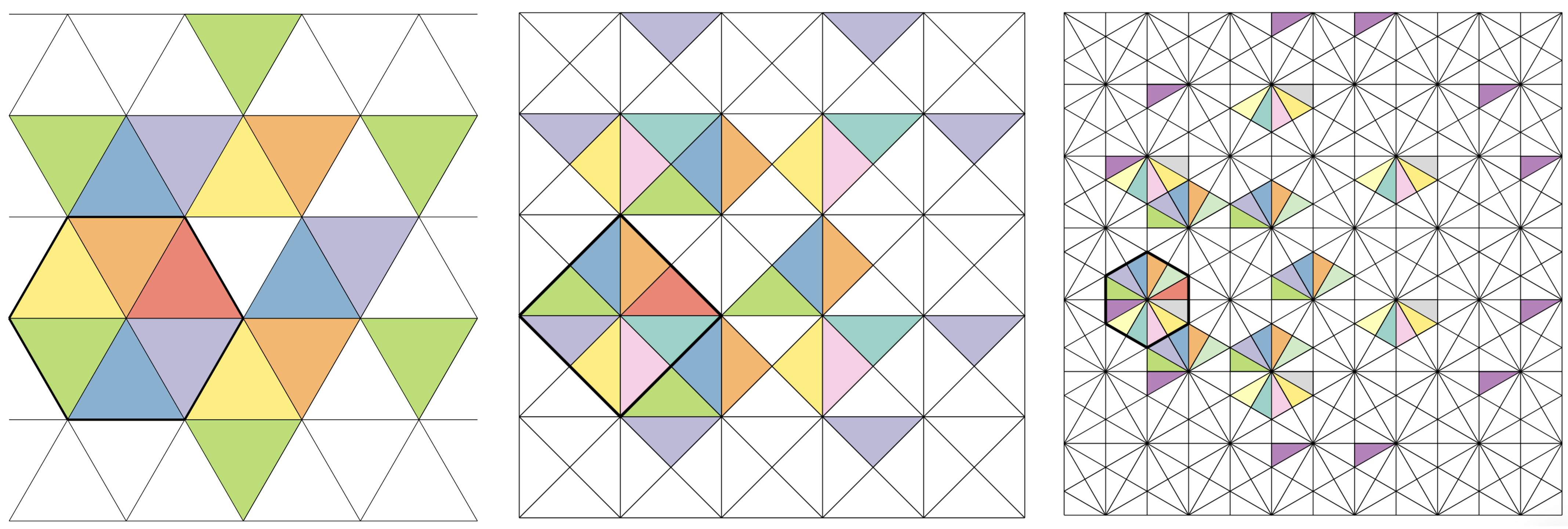
  Accordingly, simple objects in $\mathcal{A}_B$ are in bijection with pairs $(w, \overline{w'})$, where $w \in W$, and $\overline{w'}$ is an element of $P(w)\backslash W$.
- For SL_2, there are 3 simple objects: $(\mathrm{IC}_e, \mathrm{IC}_e), (\mathrm{IC}_s, 0), (0, \mathrm{IC}_s)$.

  For SL_3, there are 19.
- The Fourier-Deligne transforms Φ_w become easier to describe in the B-equivariant case, where they give an action of the Hecke category via convolution with costandard objects, whereas they are much more complicated in general: we pursue a general description in Section 4.

3. Combinatorics and the Semiinfinite Flag Variety

The following theorems concern the Kazhdan-Laumon category $\mathcal{O}$, which we call $\mathcal{A}_B$.

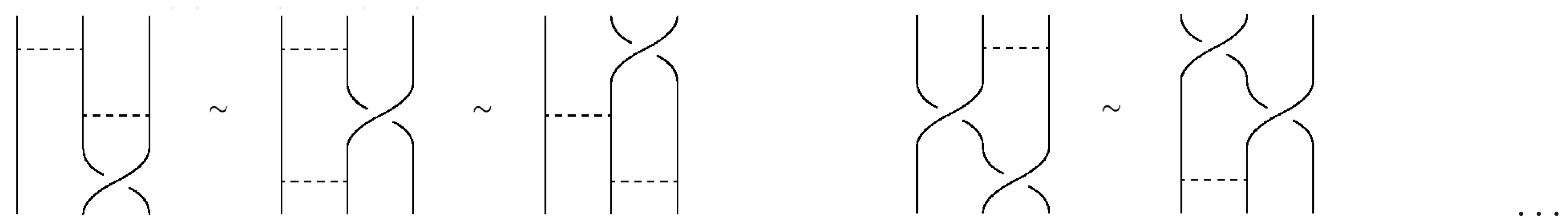
- **Theorem 2. (M-F., [3])** The category $\mathcal{A}_B$ has Grothendieck group isomorphic as a $\mathbb{C}[W] \otimes \mathcal{H}_q$-module to a submodule of *Lusztig's periodic Hecke module.*
- **Theorem 3. (M-F., [3])** This combinatorial observation can be interpreted in terms of the sheaf-function dictionary to give a connection between $\mathcal{A}_B$ and Braverman-Kazhdan's *Schwartz space* on G/U.
- **Theorem 4. (M-F., [3])** We can categorify these observations about K_0 to get an equivalence between $\mathcal{A}_B$ and a subcategory of *perverse sheaves on the semi-infinite flag variety.*



4. Braids and Symplectic Fourier Transforms

In [4], we study the symplectic Fourier-Deligne transform endomorphisms of $K_0(G/U)$ directly. Their action defines a quotient of the group algebra of the braid group, and though certain relations were understood, it was unknown how to characterize this quotient.

- There is a known algebra defined by Aicardi, Juyumaya, and Marin called the *generalized algebra of braids and ties* and a canonical subalgebra generated by the "Juyumaya generators" which can be described purely diagramatically.
- **Theorem 5. (M-F., [4])** Kazhdan and Laumon's symplectic Fourier-Deligne transform endomorphisms on G/U give a categorification of this subalgebra. Further, this gives a geometric interpretation of certain well-known facts about the algebra of braids and ties.



5. Revisiting Kazhdan and Laumon's Conjecture

- **Conjecture (Kazhdan-Laumon, [1]).** Kazhdan-Laumon categories each have finite cohomological dimension, and this fact can be used to construct a well-defined "Grothendieck-Lefschetz pairing" on the Grothendieck group. Taking the quotient by the kernel of this pairing yields a new construction of discrete series representations of $G(\mathbb{F}_q)$.
- **Proposition (Bezrukavnikov-Polishchuk, [2]).** This conjecture is false. However, a new conjecture about "rationality" of the Grothendieck group of Kazhdan-Laumon categories would be enough to salvage the attempted construction of representations.
- **Ongoing work.** Leveraging our understanding of Kazhdan-Laumon categories from above to prove Bezrukavnikov-Polishchuk's conjecture and finally provide a well-defined construction of Kazhdan-Laumon representations.

6. References

[1] D. Kazhdan and G. Laumon. Gluing of perverse sheaves and discrete series representation. *J. Geom. Phys.*, 5(1):63–120, 1988.

[2] Alexander Polishchuk. Gluing of perverse sheaves on the basic affine space. *Selecta Math. (N.S.)*, 7(1):83–147, 2001. With an appendix by R. Bezrukavnikov and the author.

[3] Calder Morton-Ferguson. Kazhdan-Laumon Category $\mathcal{O}$, Braverman-Kazhdan Schwartz space, and the semi-infinite flag variety, 2022.

[4] Calder Morton-Ferguson. Symplectic Fourier-Deligne transforms on G/U and the algebra of braids and ties, 2023.